The intensity of Aeolian tones

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SUMMARY

The generation of Aeolian tones is interpreted in terms of the theory of aerodynamic sound. To do this, the intensity of the radiated sound is expressed in terms of the fluctuations in force upon a moving rigid body using the approximation that, at low Mach numbers, these forces can be calculated assuming incompressibility of the flow. The fluctuations in lift and drag upon a circular cylinder at Reynolds numbers between 40 and 160 are calculated by integrating the fluctuations in momentum in the eddying wake, using the experimental data of Kovasznay (1949). It is found that the fluctuating lift per unit length is approximately

$$f_l = 0{\cdot}38\rho U^2 d \cos 2\pi nt,$$

where ρ is the density of the fluid and t the time, and that the magnitude of the fluctuations in drag is about 10% of this.

The axial length scale of these force fluctuations was found by an auxiliary experiment to be very large for Reynolds numbers below 100. When $100 < R < 160$, the length scale is approximately $17d$, the transition apparently occurring as a result of the instability to three-dimensional disturbances of the laminar eddying wake. Using this datum, the mean square acoustic pressure generated by the motion about the cylinder at these Reynolds numbers is found to be

$$\overline{p^2}(r) \sim 0{\cdot}27 \cos^2\theta \frac{\rho^2 l d U^6 S^2}{a^2 r^2},$$

where θ is the angle between the direction of observation and the incident stream, l the length of the cylinder and a the velocity of sound in the medium. Experiments undertaken to test this result directly, using sound intensity measurements from a whirling arm apparatus in an accoustically quiet room, gave very good agreement.

At higher Reynolds numbers, when the cylinder wake is turbulent, the theory leads to a similar expression for $\overline{p^2}$ but with a smaller numerical factor. Analysis of previous experimental data on this basis gives good agreement, with a value of about 0·037 for the numerical constant. In view of this, it is likely that some of the earlier data which had appeared to give a Mach number dependence of M^4, not M^6, have been misinterpreted.

Finally, mention is made of the conditions when the frequency of vortex generation is the same as the natural frequency of the wire, and the greatly increased intensity of the sound is seen to be the result of the greatly increased axial length scale of the force fluctuations.

The relative simplicity of this phenomenon makes possible perhaps the least ambiguous confirmation yet found of some of the essential ideas in the theory of aerodynamic sound.

1. Introduction

Aeolian tones were known in antiquity. The Aeolian harp is an old musical toy which, when placed in the wind, produces these sounds in strange successions and combinations. They are heard in the singing of the wind through telegraph wires and its sighing through the twigs and branches of a leafless tree. It was not until 1878 that the first scientific study of them was made by Strouhal, whose apparatus consisted of a cylinder mounted upon a whirling arm so that it moved through the air at constant speed. Already by 1896, Rayleigh had recognized that the production of Aeolian tones is connected with the instability of the vortex sheets in the cylinder wake, and most significantly, that it is *not* essential that the cylinder should partake of the vibration to generate these tones, although the intensity is greatly enhanced by resonance between the natural frequency of the wire and the frequency of the vortex instability. He further perceived that the frequency of the note (expressed non-dimensionally as the Strouhal number S) depends upon the Reynolds number of the flow past the cylinder, and until recently, much of the later work on this phenomenon was connected with the determination of this dependence, and with the relation between the frequency of the note and that of the vortex instability in the wake. Relf (1921) showed that the fundamental frequency of the note and the vortex frequency on each side of the cylinder wake are equal over a range of Reynolds numbers which has since been extended to values between 50 and 10^4 by the work of Kovasznay (1949), Roshko (1953) and Gerrard (1955). Measurements of the intensity of the sound under various conditions were made by Holle (1938) and Gerrard, who measured its directional distribution also. Comparison will later be made between the results of these intensity determinations and the theoretical predictions of this paper. In 1896, Rayleigh wrote concerning this problem that "a dynamical theory has yet to be given", and to some extent this statement remains true. The purpose of this paper is to interpret the properties of these tones in terms of our empirical knowledge of the flow pattern; the prediction of the latter still lies beyond the attainments of present theory.

Rayleigh's important observation that Aeolian tones can be produced without vibration of the cylinder suggests that the phenomenon can be interpreted within the framework of the theory of aerodynamic sound.

However, to include the resonant case also in our considerations, the definition of this term can be extended slightly. It has previously been restricted to sound generated by the action of aerodynamic forces upon fixed surfaces, but since in many instances, these forces will themselves produce vibrations of the surfaces, the term can conveniently be extended to include those cases in which oscillations of the surface may occur as a result of the instability of the basic flow. A distinction is therefore drawn in what seems the natural place, between such situations and those in which the vibrations are caused by some external mechanism unrelated to the flow (a violin bow, perhaps). In the latter class, it is the vibrations themselves that set up the fluctuating force in the medium, whereas in aerodynamic sound the vibrations may occur as a result of the force fluctuations from the flow.

2. The sound field

The two well-known papers by Lighthill (1952, 1954) which laid the foundations of the subject of aerodynamic sound, were concerned with the sound field generated by the unsteady motion of an unbounded fluid. More recently, Curle (1955), using the same technique as Lighthill, was able to account for the effect of solid boundaries to the flow and showed that the sound can be considered to be produced by three separate mechanisms. The first is due to the unsteadiness in the fluid itself, and as Lighthill showed, is equivalent acoustically to a distribution of quadrupoles throughout the fluid. Curle pointed out that, in general, the presence of boundaries introduces additional dipole contributions to the sound field from the fluctuating stresses acting upon the fluid at the boundary and also perhaps source contributions from dilatation of the boundaries. The general expression for the density field $\rho(\mathbf{x})$ involves terms which represent these three effects separately, and can be written as

$$\rho(\mathbf{x})-\rho_0 = \frac{1}{4\pi a_0^2}\frac{\partial^2}{\partial x_i \partial x_j}\int \frac{1}{r}\,T_{ij}(\mathbf{y})\,dV(\mathbf{y}) + \\ + \frac{1}{4\pi a_0^2}\frac{\partial}{\partial x_i}\int \frac{1}{r}(\rho v_i v_j + p_{ij})\,dS_j(\mathbf{y}) - \frac{1}{4\pi a_0^2}\int \frac{1}{r}\frac{\partial(\rho v_i)}{\partial t}\,dS_i(\mathbf{y}), \quad (2.1)$$

where ρ_0 and a_0 are the density and velocity of sound in the undisturbed medium, v_i and p_{ij} are the velocity and stress tensor at a point $\mathbf{y}$ in the fluid and $r = |\mathbf{x}-\mathbf{y}|$. The tensor T_{ij} is given by

$$T_{ij} = \rho v_i v_j + p_{ij} - a_0^2 \rho\, \delta_{ij}, \quad (2.2)$$

and the integrands in (2.1) are taken at the retarded time $t - r/a_0$.

This gives the fluctuations in density in the medium in terms of the properties of the flow, which are assumed to be specified *a priori*. In general, of course, the density field itself must be included in the specification of the flow, and in practice (2.1) is meaningful only to the first order in the Mach number in which the direct dynamical interaction between the sound field and the velocity field can be ignored. To obtain this first order term, it is legitimate to substitute into the right-hand side of (2.1) the values that would obtain if the fluid were incompressible, and this, incompressibility approximation' has invariably been made in the applications of the theory up to the present time. With its aid, retaining the leading term in each of the integrals, it is found that the radiated part of the sound field from the motion in the neighbourhood of a finite closed body is given by

$$\rho(\mathbf{x})-\rho_0 \sim \frac{1}{4\pi a_0^4}\frac{x_i x_j}{x^3}\int \frac{\partial}{\partial t^2} T_{ij}\, dV(\mathbf{y}) + \frac{1}{4\pi a_0^3}\frac{x_i}{x^2}\int \frac{\partial}{\partial t}(\rho_0 v_i v_j + p_{ij})\, dS_j(\mathbf{y}) - \\ - \frac{1}{4\pi a_0^2}\frac{\rho_0}{x}\int \frac{\partial v_i}{\partial t}\, dS_i(\mathbf{y}). \quad (2.3)$$

Curle (1955) simplifies this expression when the boundaries are perfectly rigid—perfect acoustic reflectors—and at rest, so that $v_i \equiv 0$ over $S_i(\mathbf{y})$, but it will be shown here that the same simplification is possible in rather less restrictive circumstances. A sufficient condition for the vanishing of the source contribution is simply that the body be incompressible, so that its dilatation is zero. The Reynolds stress term in the dipole contribution also vanishes if the surface moves uniformly as that of a rigid body. For then $v_i(\mathbf{y}) = u_i + \epsilon_{ijk}\omega_j y_k$ on $S_i(\mathbf{y})$, where u_i is the velocity of, and ω_i the angular velocity about, the centre of mass, and on substitution into this term of the integral and application of Gauss's theorem, it is found to vanish. This second condition clearly embraces the first and (2.3) reduces to

$$\rho(\mathbf{x})-\rho_0 \sim \frac{1}{4\pi a_0^4}\frac{x_i x_j}{x^3}\int \frac{\partial^2 T_{ij}}{\partial t^2}\, dV(\mathbf{y}) - \frac{1}{4\pi a_0^3}\frac{x_i}{x^2}\frac{dF_i}{dt}, \quad (2.4)$$

where $F_i = -\int p_{ij}\, dS_j$ is the total force with which the body acts upon the fluid. Finally, when the orders in the Mach number M of the two terms on the right of this equation are examined, it is found that the dipole contribution is of order M^3 whereas the quadrupole term is of order M^4, that is, of the same order as the quantities already neglected in the incompressible approximation to the dipole term. So the quadrupole term too can be neglected and, when M is small,

$$\rho(\mathbf{x})-\rho_0 \sim -\frac{1}{4\pi a_0^3}\frac{x_i}{x^2}\frac{dF_i}{dt}. \quad (2.5)$$

In the generation of Aeolian tones, the fluctuating force is distributed along the length of the cylinder, and at low Reynolds numbers, if the flow were accurately two-dimensional, the force fluctuations would be in phase at all points. However, there are many reasons for believing that in a practical situation with a long cylinder (say, a long, thin wire) this will not be so, but that the phase of the force fluctuations will vary slowly, probably randomly, along its length. One reason is that the inherent small variations in the cylinder diameter will result in small changes in the frequency of vortex generation at different points which, in a large number of cycles, produce finite variations in phase. The coupling between the motions in planes separated by a large number of diameters cannot be presumed to be so tight that such phase differences will not occur. Again, even with a perfectly uniform cylinder, if there is a three-dimensional instability of any kind, phase differences assuredly will be found, the extreme case being given when the flow behind the cylinder is turbulent. This question will be considered later in some detail, but if, for whatever reason, a random phase variation is present, then the sound intensity radiated from a cylinder of length l is approximately

$$I(\mathbf{x}) \sim \frac{x_i x_j}{16\pi^2 \rho a^3 x^4} \, l \int_{-\infty}^{\infty} \overline{\frac{\partial f_i(z_0)}{\partial t} \frac{\partial f_j(z_0+z)}{\partial t}} \, dz,$$
$$\sim \frac{1}{16\pi^2 \rho a^3 x^4} \overline{\left(x_i \frac{\partial f_i}{\partial t}\right)^2} \, sdl, \qquad (2.6)$$

where $f_i(z)$ is the fluctuation in force per unit length at the axial position z, s is the dimensionless correlation distance along the axis (so that the actual distance is sd) and the subscripts to ρ and a have been omitted. In (2.6) it is further assumed that $l \ll sd$, as seems likely in many practical cases, so that end effects can be neglected. The two principal quantities to be determined on the right of this expression are first the mean square value of $\partial f_i/\partial t$ and second the axial scale s. The first is to be found from the experimental results of Kovasznay (1949) for the eddying motion in the wake at a Reynolds number in the regime of two-dimensional flow. This restricts our consideration initially to Reynolds numbers below those for which the motion is turbulent, although the theory is extended in a later section to conditions beyond this range. The second quantity, the correlation distance s, is the subject of a subsidiary experiment described in § 5.

3. The fluctuations in lift and drag upon a circular cylinder

The fluctuations in lift and drag have been calculated by integration of the acceleration field of the eddying wake in the directions normal and parallel to the direction of the oncoming stream, using the very extensive experimental data given by Kovasznay (1949) for a Reynolds number of 50. He expressed the velocity fluctuations in the direction of the oncoming stream, in the two-dimensional motion, as

$$u(x,y,t) = \phi_1(x,y)\cos 2\pi[\zeta_1(x)+nt] + \phi_2(x,y)\cos 4\pi[\zeta_2(x)+nt], \qquad (3.1)$$

where the amplitude functions ϕ_1 and ϕ_2 vary slowly with x (the downstream coordinate), but rapidly with y, i.e. across the stream. ϕ_1 is an odd function of y, and ϕ_2 an even function. Here $\zeta_1(x)$ and $\zeta_2(x)$ are not quite linear functions of x, in order to account for the slightly non-uniform spacing of the vortices, and the frequency n is constant in time. Kovasznay measured $\phi_1(x,y)$ and $\phi_2(x,y)$ across the wake at various stations downstream as far as $x/d = 40$, and, by smoothing and interpolating between his points, these functions have been found for values of y within the wake and for values of x between 0 and $40d$. Interpolated values of the functions $\zeta_1(x)$ and $\zeta_2(x)$ have also been obtained from his measurements.

The fluctuations in drag per unit length can be found directly by integration of (3.1) and use of the equation

$$f_d = \rho \frac{d}{dt} \iint u \, dxdy,$$

where the integration is over a plane normal to the cylinder axis. The calculation presents no difficulties concerning convergence, since $\phi_2(x,y)$ is negligibly small for values of y beyond the centres of the vortices. The function $\phi_1(x,y)$ is odd in y, so that the integral of the first term in (3.1) vanishes, and the fluctuations in drag have a frequency just twice the fundamental n. However, the calculation of the fluctuations in lift per unit length f_l is not so straightforward, since the velocity fluctuations normal to the oncoming stream were not measured. They could perhaps be calculated from the continuity condition

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

which gives

$$v = \int_{\infty}^{y} \frac{\partial u}{\partial x} \, dy.$$

The difficulty with this method is that measurements of u were not made beyond the vortical region of the wake, and to make a correct allowance for the virtual mass of the irrotationally moving fluid would indeed be complicated. An alternative method was used which required information about the vorticity field alone.

It is shown in the Appendix that

$$\iint v \, dxdy = -\iint \omega x \, dxdy, \tag{3.2}$$

where the vorticity ω, which vanishes outside the wake, is given by

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

so that, using the continuity condition,

$$\frac{\partial \omega}{\partial y} = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}.$$

Hence

$$\omega = \int^{y} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dy, \tag{3.3}$$

the integration being commenced at the edge of the wake. The integral $\iint v \, dxdy$ is calculated from (3.2), (3.3) and (3.1), and some of the details of this, including the devices used to ensure more rapid convergence, are described in the Appendix.

The result is found to be

$$M = \rho \iint v \, dxdy = 0{\cdot}48 \rho U d^2 \cos 2\pi n t, \tag{3.4}$$

where U is the mean stream velocity, and thus the fluctuating lift per unit length is

$$f_l = \frac{dM}{dt} = 0{\cdot}38\rho U^2 d \cos 2\pi nt, \tag{3.5}$$

(taking a new time origin), since $n = 0{\cdot}13U/d$ at $R = 50$. The fluctuation in drag per unit length is found by direct integration, and its magnitude is very much smaller. In fact,

$$f_d = 0{\cdot}04\rho U^2 d \cos 4\pi nt. \tag{3.6}$$

Just as the fluctuations in drag are determined by the term in (3.1) which is an even function of y, so the fluctuating lift is determined by the odd part in y, as the result (3.5) implies. For, from (3.2), only the even part of ω is relevant, or, from (3.3), only the odd part of u.

The mean drag per unit length at a Reynolds number of 50 is about $0{\cdot}75\rho U^2 d$, so that our calculation shows that the magnitude of the maximum lift is about 50% of the mean drag.

The value 0·38 for the coefficient in (3.5) can be shown to be a not unreasonable figure by constructing a very crude upper limit to the amplitude of the fluctuating lift in the following way. Suppose that vortex sheets of strength U are generated at each side of the cylinder which, because of their instability, curl up into discrete vortices like a Karman street. Suppose further that all the vorticity is gathered into these vortices, none being lost by diffusion. The separation of the vortices is approximately $\frac{1}{2}U/n$, which, at a Reynolds number of 50 and Strouhal number of 0·13, is equal to nearly $4d$. The strength of the discrete vortices is therefore $4Ud$, and the amplitude of the fluctuating circulation about a fixed contour in the irrotational fluid is $2Ud$, so that the amplitude of the fluctuating lift is $\rho U \,.\, 2Ud = 2\rho U^2 d$. That this is so much greater than the actual value can presumably be ascribed to the fact that the process of viscous diffusion would lead to a reduction of the coefficient.

The accuracy of the calculated value (3.5) is admittedly not very high. The numerical differentiations reduce the accuracy, only partial compensation being provided by the subsequent integrations, and inspection of the working suggests that an accuracy of 40% is probably the best that can be claimed. However, this is ample for the present purpose in view of the uncertainties of the acoustical measurements themselves.

4. The effect of Reynolds number

The estimate (3.5) was obtained from consideration of the flow pattern at $R = 50$, and it would be very satisfying if similarly detailed measurements were available at other Reynolds numbers in the range of two-dimensional eddying motion. Unfortunately, they are not, and some general argument will have to be invoked concerning the effect of viscosity upon the flow pattern. Such an argument would not be required to give results that are accurate in detail, but only to give a good indication of the variation of the amplitude of the lift fluctuations with Reynolds number.

Let us consider the mechanism of the development of the vortices in the wake. The initial instability introduces small fluctuations in the y-position of the vortex layer on each side of the wake, and these result in differential convection velocities of the vorticity and the eventual rolling up of the layer. The development of the vortices is opposed by the diffusive action of viscosity, and the maximum concentration is reached when the two effects balance. Thereafter the vortices decay by viscous diffusion between those of opposite sign, but it is not difficult to show that, at $R = 50$, this process hardly commences within the first 50 diameters (this being the part of the wake upon which the lift and drag calculations of § 3 depend). For, in time t, the vorticity diffuses over a distance of order $(\nu t)^{1/2}$, so that at a point k diameters downstream it has diffused a distance of order $(\nu kd/U)^{1/2} = (k/R)^{1/2}d$. When $R = 50$, this is of order d (still much less than the vortex spacing) at 50 diameters from the cylinder; the details of the motion in the wake beyond such points having little influence upon the fluctuating forces on the cylinder. For the present purposes, then, a decrease in the viscosity of the fluid (or an increase in R) has two consequences; it modifies the frequency of the initial instability and also postpones the attainment of the balance between viscous and convective vorticity transfer, so that the vortices become more concentrated.

Regarding the second effect, (3.2) shows that, in discussing the fluctuating lift, we are concerned not so much with the concentration of the fluctuations in vorticity as with their integrated magnitude or with the total strength of the vortices. If we assume as an approximation that the lateral spacing of the vortices is materially unaltered by a decrease in viscosity, the Reynolds number effect must lie in the variation of the strength of the vortices. This is indeed a consequence of the first effect, the modification of the frequency of the initial instability and so of the spacing of the vortices. The strength of the vortices can be assumed to be proportional to their spacing, U/n approximately, so that as the viscosity varies, the vortex strength varies as S^{-1}, since $S = nd/U$. Therefore the momentum fluctuations normal to the direction of the oncoming stream are proportional to S^{-1}, and on differentiation f_l is independent of S and so of R; consequently

$$f_l \simeq 0{\cdot}38\rho U^2 d\cos 2\pi nt, \tag{4.1}$$

for $40 < R < 160$. It follows immediately that

$$\frac{\partial f_l}{\partial t} = 2{\cdot}4\rho U^3 S\cos 2\pi nt. \tag{4.2}$$

The Strouhal number S is well defined empirically as a function of Reynolds number (Kovasznay 1949, Gerrard 1955) so that this equation gives approximately the dependence on Reynolds number of the term $\partial f_l/\partial t$ in the expression (2.6).

5. The correlation distance s

Very little published information is available concerning the scale of the variations in phase of the fluctuations along the axis of the cylinder. Both Roshko (1953) and Gerrard (1955), making observations at Reynolds numbers greater than about 150, found slow spanwise variation in phase along the cylinder, but insufficient data were available to estimate its length scale. On the other hand, Kovasznay (1949) found no measurable differences at $R = 50$ over distances of the order of 20 diameters, so that presumably the length scale is very much greater than this. An estimate of s is necessary for the application of (2.6), and so to obtain this, and to clarify the situation if possible, a supplementary experiment was made in a water tank.

A hollow tube of outside diameter $\frac{1}{4}$ in. was stopped up at one end and a row of small holes drilled down one side at $\frac{1}{4}$ in. centres. These holes were then filled with plasticine which could be removed easily from two of them to give open holes separated by any desired number of diameters from 1 to 30. The tube was filled with dye and lowered vertically into a tank of still water. The tube was drawn through the water with a given velocity, and the vortex patterns were observed at two different axial positions by the dye traces at the two free holes. End effects were avoided by making no observations within 10 diameters of the stopped end of the tube or within 4 diameters of the free surface. The differences in phase were measured by looking vertically downwards at the dye patterns behind the cylinder, and, by repeating the experiment a large number of times for points at a given separation, the correlation coefficient of the fluctuations in lift was estimated. By repeating this procedure with holes at different separations, the length scale of the lift fluctuations was determined approximately.

The results of these observations can be summarized as follows. For Reynolds numbers below about 80, very little phase difference was observed even for the largest available separation of 30 diameters, so that under these conditions the axial scale is certainly much greater than this. However, when the Reynolds number was between 100 and 160, the phase differences were clearly observed. At a separation of about 30 diameters, the phases appeared to be almost uncorrelated. The scale s was estimated to be between 15 and 20 diameters, and did not appear to depend critically upon Reynolds number or upon the presence or absence of small disturbances in the water before the passage of the cylinder. At Reynolds numbers between these two ranges, i.e. between 80 and 100, it was noticed that there was little difference in phase if the tank was allowed to settle for a minute or two between successive trials, but that random variations were set up with about the same scale as at higher Reynolds numbers if two trials were carried out in rapid succession or if the water was disturbed slightly prior to an attempt.

These observations suggest that, at Reynolds numbers between about 40 and 80, the eddying motion is truly two-dimensional, and apparently stable to small disturbances in the oncoming stream. As the Reynolds number increases to a value between 80 and 100, however, the laminar motion becomes unstable to three-dimensional disturbances with a length scale along the axis of about $17d$, and the flow becomes only approximately two-dimensional. This new type of motion persists until about $R = 160$, when yet further modes are excited and the flow eventually becomes turbulent. These observations are consistent with those already mentioned for Reynolds numbers of 50 and 150.

Returning to our basic problem, it seems that a reasonable value for s in (2.6) is about 17 for Reynolds numbers between 100 and 160. Below this range, a much larger value can be expected, and its magnitude would be determined, not by the flow conditions, but by the small imperfections and irregularities in the diameter of the wire, as has been explained in § 2. These can hardly be considered to be under the control of the experimenter, and reasonable *a priori* estimates of the magnitude of the radiated sound are difficult without detailed examination of each particular wire. One further point arises from these observations. The fluctuating lift and drag were calculated on the assumption of a two-dimensional flow, and some error will be introduced into (4.1) by the lack of two-dimensionality at values of R between 100 and 160. However, since the scale of the variations along the axis of the cylinder is very much greater than that in the plane normal to the axis, such errors are expected to be quite small.

6. Comparison with experiment

The radiated intensity can be found immediately from (2.6) and (4.1). Neglecting the relatively small fluctuations in drag, the radiated sound is generated by the fluctuating lift, and so consists of a dipole field with its axis normal to the direction of the oncoming stream. From (4.2),

$$\overline{\left(\frac{df_l}{dt}\right)^2} = 2{\cdot}4\rho^2 U^6 S^2,$$

so that the total radiated intensity at points distant r from a cylinder of length l is

$$I(r) \sim 0{\cdot}27 \cos^2\theta \frac{\rho l d U^6 S^2}{a^3 r^2}, \tag{6.1}$$

where θ is the angle between the oncoming stream and the direction of the line joining the point of observation to the cylinder, and s has been taken as 17. The mean square acoustic pressure is

$$\begin{aligned} \overline{p^2}(r) &= \rho a I(r), \\ &\sim 0{\cdot}27 \cos^2\theta \frac{\rho^2 l d U^6 S^2}{a^2 r^2}. \end{aligned} \tag{6.2}$$

Some experiments were undertaken to test this result, particularly the variation of $\overline{p^2}$ as the sixth power of the Mach number. The measurements of sound intensity, which were made at the Acoustics Laboratory of the

National Physical Laboratory, Teddington, by the kind permission of the Superintendent of the Physics Division and Mr N. Fleming, were executed by Mr W. C. Copeland. I am greatly indebted to them for their generous and able assistance. The apparatus consisted of two thin wires, 60 cm in length and 0·0123 cm in diameter, held vertically with just sufficient tension to prevent sagging, between two arms mounted on a spinning shaft of diameter $\frac{5}{8}$ in.—the same type of arrangement as that used by Strouhal and Gerrard. The supporting arms were made of $\frac{3}{16}$ in. dural sheet tapered and streamlined to reduce disturbances. The shaft was belt-driven from an electric motor, and, to reduce the background noise level, the motor and all the ball-bearings were enclosed in felt-lined boxes. The apparatus was mounted on an open steel frame and, when in position for the measurements, was placed in the acoustically quiet room at the N.P.L. with a microphone 115 cm from the axis of the shaft in the plane bisecting the wires. The apparatus was tested by running it with the wires removed, and the noise level was found to be negligible in the frequency range at which the measurements were made.

The microphone signal was amplified and passed through a variable frequency narrow-band analyser, and displayed on the screen of a cathode-ray oscillograph. The modulation with time, corresponding in the whirling arm apparatus to the variation with θ, followed closely that indicated by the $\cos^2\theta$ factor in (6.2). For the intensity measurements, the peak values of $\overline{p^2}$, corresponding to orientations $\theta = 0$, were observed at a number of frequencies within the narrow spectrum of the sound generated, from which the total radiation was calculated. The speed of the wires was altered by variation of their radial position and of the speed of the motor. The results of these measurements are shown as the solid points in figure 1, together with the predicted relation (6.2), and the agreement is seen to be remarkably good. However, the constant in (6.2) is reliable only to within a factor of two, this range of uncertainty being approximately the same as the experimental uncertainty shown by the scatter of the points. For values of $U(ldS^2/r^2)^{1/6}$ less than about 140, corresponding in these measurements to Reynolds numbers less than about 100, an intermittent signal of very much higher intensity was sometimes recorded, which was consistent with the establishment of the regime of truly two-dimensional flow over most of the length of the wires and the consequent enormous increase in the value of s. However, under the experimental conditions, it was not possible to produce a sufficiently stable signal to make systematic observations, and indeed, even had this been possible, there remains the strong suspicion that the results would not be generally applicable, but would depend upon the imperfections of the particular wire used.

These results offer good evidence for the validity of the expression (6.2) in the Reynolds number range from 100 to 160. The variation of the sound intensity as the sixth power of the velocity seems to be confirmed well, and in the next section it will be shown that this remains true when the flow is turbulent at higher Reynolds numbers.

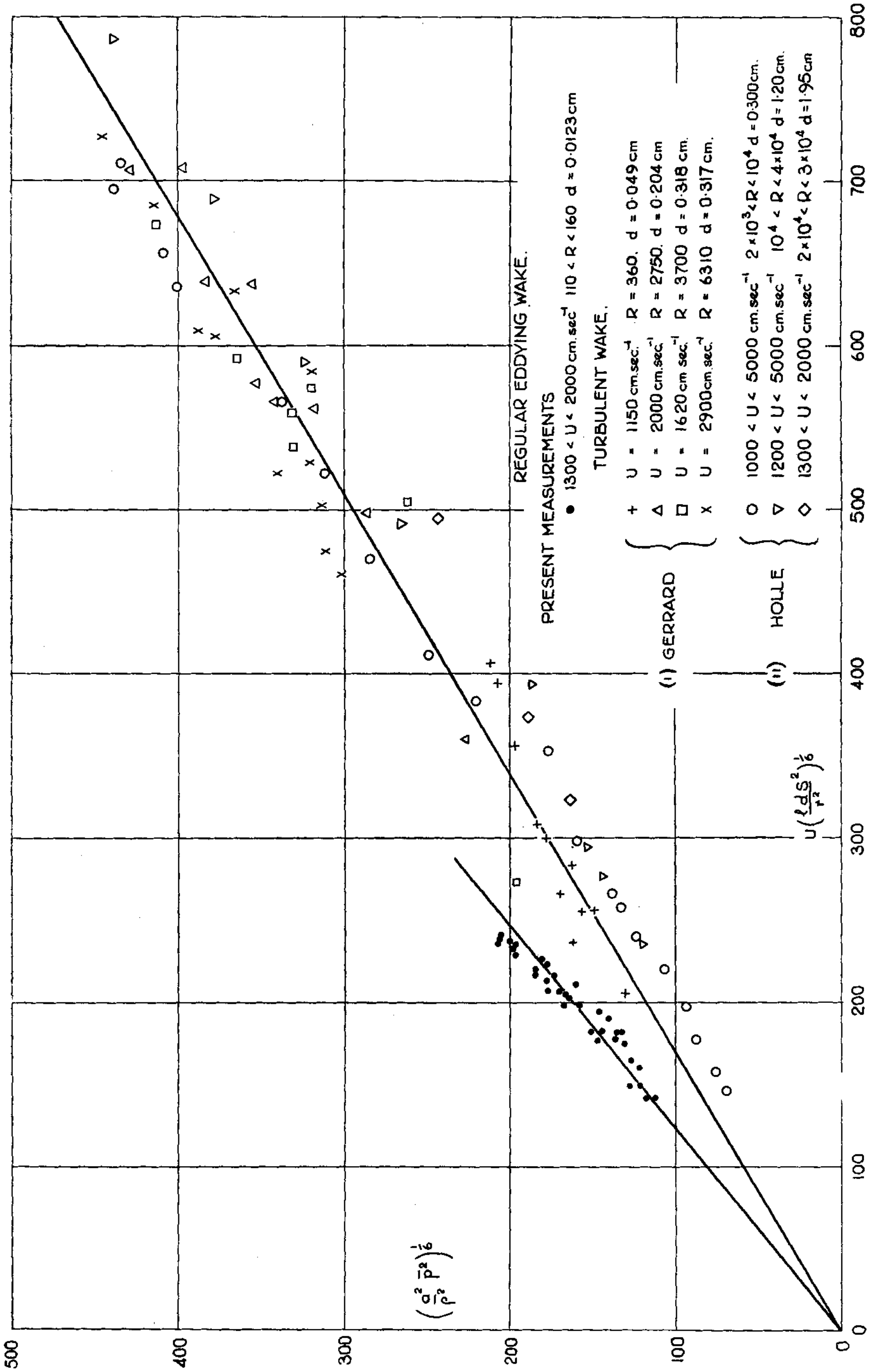
REGULAR EDDYING WAKE.
PRESENT MEASUREMENTS
● $1300 < U < 2000$ cm.sec^{-1} $110 < R < 160$ $d = 0{\cdot}0123$ cm
TURBULENT WAKE.
(i) GERRARD
+ $U = 1150$ cm.sec^{-1} $R = 360$. $d = 0{\cdot}049$ cm
△ $U = 2000$ cm.sec^{-1} $R = 2750$ $d = 0{\cdot}204$ cm
□ $U = 1620$ cm sec^{-1} $R = 3700$ $d = 0{\cdot}318$ cm
× $U = 2900$ cm.sec^{-1} $R = 6310$ $d = 0{\cdot}317$ cm
(ii) HOLLE
○ $1000 < U < 5000$ cm.sec^{-1} $2\times10^3 < R < 10^4$ $d = 0{\cdot}300$ cm
▽ $1200 < U < 5000$ cm.sec^{-1} $10^4 < R < 4\times10^4$ $d = 1{\cdot}20$ cm
◇ $1300 < U < 2000$ cm.sec^{-1} $2\times10^4 < R < 3\times10^4$ $d = 1{\cdot}95$ cm
$\left(\frac{a^2}{\rho^2}\bar{p}^2\right)^{\frac{1}{6}}$
$U\left(\frac{\ell d S^2}{r^2}\right)^{\frac{1}{6}}$
0
100
200
300
400
500
600
700
800

Figure 1.

7. Turbulent flow

Hitherto attention has been concentrated upon conditions at Reynolds numbers below that for the onset of turbulent flow. As R increases beyond this range, intermittent bursts of turbulence occur in the flow, as described by Roshko (1953), until at about $R = 300$ the wake behind the cylinder is fully turbulent. The expressions given for the fluctuating lift and axial length scale are no longer relevant but we can still predict their order of magnitude and dependence on Reynolds number and obtain, for these new conditions, an expression for the intensity of the radiated sound which is arbitrary only to the extent of an unknown constant factor.

It is clear that the presence of a fluctuating lift depends upon the existence of large-scale axial eddies of alternating sign near the cylinder. Some measurements have recently been made by Mr I. C. T. Nisbett at the Cavendish Laboratory in the turbulent flow near the cylinder at Reynolds numbers of order 5×10^3. He has established the presence of a surprisingly distinct large-scale structure, similar in many ways to that observed in the two-dimensional eddying flow at lower Reynolds numbers. These large eddies are approximately parallel to the cylinder axis, and contain an appreciable amount (about half) of the turbulent energy, the remainder being in the irregular smaller-scale fluctuations. They appear to be somewhat contorted in the axial direction, since simultaneous velocity correlations between similar points in planes normal to the axis are alternately positive and negative as the axial separation increases up to about 20 diameters. The integral length scale of these eddies along the axis is about three diameters. These observations, in the light of our previous discussion and of such formal expressions as (3.2), suggest that the fluctuations in lift are still dominant even at these Reynolds numbers, and that their length scale is about $3d$, although we can make no *a priori* statement about their magnitude.

What of the variation of the magnitude of the fluctuating lift with Reynolds number? Here, again, the general arguments of § 4 still apply with equal force, and we have

$$(\overline{f_l^2})^{1/2} = \text{constant}\ \rho U^2 d. \tag{7.1}$$

Some further support to this relation is afforded by the principle of Reynolds number similarity (Townsend 1956). This principle expresses the fact that the large-scale components of the motion are not influenced by the fluid viscosity if it is sufficiently small, i.e. by the Reynolds number provided it is sufficiently large. We have seen that the fluctuating lift is a consequence of the large eddy motion in the wake, so that (7.1), which is correct dimensionally, is also independent of the (large) Reynolds number. Whether or not the principle of Reynolds number similarity holds in this situation for values of R as low as 300 is perhaps open to question, but for higher values it can certainly be invoked to support (7.1). Using this relation, then, we have from (2.6) that the radiated intensity from a cylinder of length l is

$$I(r) = \alpha \cos^2\theta \frac{\rho l d U^6 S^2}{a^3 r^2}, \tag{7.2}$$

which differs from (6.1) only in the constant factor which is now represented by α. Again, as in (6.2), the mean square acoustic pressure is given by

$$\overline{p^2}(r) = \alpha \cos^2\theta \frac{\rho^2 l d U^6 S^2}{a^2 r^2}, \tag{7.3}$$

where the Strouhal number S involves the frequency of 'shedding' of the large eddies, or the fundamental frequency of the sound generated. The constant α can be expected to be considerably smaller than 0·27, because of the reduced length scale of the fluctuations in lift per unit length and also possibly a reduced intensity arising from the decreased coherence in the turbulent flow.

Detailed measurements of the sound intensity radiated at Reynolds numbers for which the flow is turbulent have been made by Holle (1938) and by Gerrard (1955). Now, Gerrard's curves of intensity as a function of velocity at constant Reynolds number (obtained by keeping the product Ud fixed) appear to show that $\overline{p^2} \propto U^4$, but this behaviour is inconsistent with the U^6 variation suggested by a rough application of the theory (as he himself points out) and his results have hitherto presented something of an anomaly. However, (7.3) shows that, with Ud fixed, we should expect that $\overline{p^2} \propto U^5$, so that the anomaly is not as serious as might appear at first. A possible cause of the discrepancy lies in Gerrard's having taken $\overline{p^2} \propto l^2/r^2$ when $l \gg d$ in the analysis of his data, which implies that the fluctuations in lift are in phase along the entire length of the cylinder, over many hundreds of diameters. In turbulent flow, the case for assuming a random phase variation along the cylinder (leading to intensities proportional to the *first* power of the length as in (7.2)) is even stronger than at the lower Reynolds numbers discussed in the previous sections, yet independent confirmation that this is so even under these latter conditions has been given by the agreement between the absolute values of the measured and predicted sound intensities shown in the upper curve of figure 1*. Nisbett's observations on the flow at high Reynolds numbers provide yet further strong evidence for random phase variations, so that it seems inevitable that, in fact, $\overline{p^2} \propto l/r^2$ in the radiated field under the circumstances of Gerrard's measurements.

In his figure 4, some of Gerrard's measured values of $\overline{p^2}$ under a variety of conditions are plotted as a series of curves together with the relevant experimental parameters. Assuming that in fact $\overline{p^2} \propto l/r^2$, these points have been replotted as a function of velocity and Reynolds number in the lower curve of figure 1 of this paper. This diagram also shows a

* If, in (6.2), sd is replaced by the length l, the expression would give values of $\overline{p^2}$ about 250 times those measured in the experiments of § 5.

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representative sample of Holle's (1938) measurements*, and these agree well with Gerrard's replotted points. They all lie about a single straight line passing through the origin, as predicted by equation (7.3), and cover a fairly wide range of conditions. The slope of the line is less than under conditions of non-turbulent flow and corresponds to a value of α of approximately 0·037. The scatter of the points is not inconsiderable, but the linearity of the relation seems to be confirmed well, and provides good support for the result (7.3) which is based upon the ideas of the theory of aerodynamic sound.

8. Some final remarks

The satisfactory agreement between these various experimental results and the theory gives confidence that the latter is firmly founded. One factor ignored in our considerations, however, is the possibility of vibration of the wire. However it is not difficult to show that the amplitude of the vibrations is indeed small compared with the wire diameter provided the tension is such that the natural frequency of the wire is very far from the vortex frequency, and the density of the medium is much less than that of the wire. Under these conditions, which have been satisfied in the experiments mentioned here, the vibrations of the wire can reasonably be neglected. On the other hand, if the frequency of vortex shedding coincides with the fundamental frequency of the wire, or with one of its low harmonics, then resonance will lead to vibrations of finite amplitude. Rayleigh's observation that the acoustic intensity is greatly enhanced under these conditions was mentioned earlier in this paper, and there arises the question of accounting for this within the framework of the present theory.

Now, the two basic variables which determine the intensity of the radiated sound are the magnitude of the fluctuations in lift, and the correlation distance s. The first quantity will clearly be modified by the presence of vibrations, both through the introduction into equations such as (3.2) of an additional term describing the motion of the boundary and through the modification of the flow pattern, but since the maximum velocity of the vibrating wire could not be expected to be much greater than the wind velocity, these could not reasonably be expected to contribute a factor of more than about 5 to the intensity of the sound radiation. A much more powerful contribution will be given by the increase in the value of s brought about by the equalization of the phase of the fluctuating lift along the wire. The vibrations, which may be in phase over the whole length and are at the same frequency as the vortex 'shedding', clearly provide a coupling between the motions in the fluid at different axial positions that is far more effective than any hydrodynamic coupling. The phase of the large eddies loses its randomness along the cylinder, and is determined by the phase of the vibrations, so that for the fundamental the correlation distance sd now becomes the length l, which can represent an increase of

* Incidentally, Holle found experimentally that $\overline{p^2} \propto l/r^2$ approximately, as in (7.3), and that the velocity dependence was as U^7, nearly. But a reasonably fit is still obtained by plotting $(\overline{p^2})^{1/6}$ against U as in figure 1.

many hundreds of times. The explanation of Rayleigh's observation seems to lie in this effect rather than in any increase in the magnitude of f_l.

The value of this work may go beyond the interpretation and understanding of the phenomenon of Aeolian tones. The relative simplicity of the fluid motion makes possible what is perhaps the least ambiguous confirmation yet found of some of the essential ideas in the theory of aerodynamic sound, and gives even greater confidence in their application to more complex problems.

Appendix. Fluctuations in wake momentum

From the vector identity

$$\mathbf{x} \times \boldsymbol{\omega} = \nabla(\mathbf{x} \cdot \mathbf{v}) - (\mathbf{v} \cdot \nabla)\mathbf{x} - (\mathbf{x} \cdot \nabla)\mathbf{v}, \tag{A.1}$$

it can readily be shown that

$$\int \mathbf{v}\, d\mathbf{x} = \tfrac{1}{2} \int (\mathbf{x} \times \boldsymbol{\omega})\, d\mathbf{x} + \tfrac{1}{2} \int \mathbf{x} \times (d\mathbf{S} \times \mathbf{v}), \tag{A.2}$$

where the integration is throughout the region bounded by the closed surface S. We apply this to the two-dimensional flow about a cylinder centred on the z-axis, considering the volume bounded by the planes $z = 0$ and $z = d$, say. The cylinder is considered to be fixed, and the surface of integration reduces to the two planes, so that (A.2) becomes

$$d \iint \mathbf{v}\, dxdy = \tfrac{1}{2} d \iint (\mathbf{x} \times \boldsymbol{\omega})\, dxdy + \tfrac{1}{2} \left[\int \mathbf{x} \times (d\mathbf{S} \times \mathbf{v}) \right]_{z=d} + \tfrac{1}{2} \left[\int \mathbf{x} \times (d\mathbf{S} \times \mathbf{v}) \right]_{z=0}$$

In the two-dimensional motion the fluctuations in velocity and vorticity are $\mathbf{v} = (u, v, 0)$ and $\boldsymbol{\omega} = (0, 0, \omega)$, and on substituting these into the last expression, we obtain for the fluctuations in momentum in the y-direction normal to the oncoming stream,

$$d \iint v\, dxdy = -\tfrac{1}{2} d \iint \omega x\, dxdy + \tfrac{1}{2} d \iint v\, dxdy,$$

since $d\mathbf{S} = (0, 0, -dxdy)$ on $z = d$. Therefore

$$\iint v\, dxdy = - \iint \omega x\, dxdy, \tag{A.3}$$

and similarly

$$\iint u\, dxdy = \iint \omega y\, dxdy. \tag{A.4}$$

where the limits are understood to be taken in the sense

$$\lim_{x\to\infty} \left(\lim_{y\to\infty} \int_{-x}^{x} \int_{-y}^{y} v\, dydx \right).$$

In the eddying wake behind a cylinder (of unit diameter) we shall neglect fluctuations in the vorticity of the boundary layer, since Kovasznay's measurements indicate that the unsteady motion is small within the first few diameters of the centre of the cylinder. The fluctuations in vorticity are found from equation (3.3), and on considering only that part of u in (3.1) that is an odd function in y,

$$\frac{\partial^2 u}{\partial x^2} = \left\{\frac{\partial^2 \phi_1}{\partial x^2} - 4\pi^2\phi_1[\zeta_1'(x)]^2\right\}\cos 2\pi[\zeta_1(x)+nt] -$$

$$-\left\{4\pi\frac{\partial \phi_1}{\partial x}\zeta_1'(x) + 2\pi\phi_1\zeta_1''(x)\right\}\sin 2\pi[\zeta_1(x)+nt],$$

$$= A(x,y)\cos 2\pi[\zeta_1(x)+nt] - B(x,y)\sin 2\pi[\zeta_1(x)+nt],$$

say. Therefore

$$\omega = \left(\alpha - \frac{\partial \phi_1}{\partial y}\right)\cos 2\pi[\zeta_1(x)+nt] - \beta\sin 2\pi[\zeta_1(x)+nt],$$

where $$\alpha(x,y) = \int_y^5 A(x,y)\,dy, \qquad \beta(x,y) = \int_y^5 B(x,y)\,dy,$$

the upper limit being taken at $y = 5$ since it was found that the vorticity was zero along this line over the length of the wake to be considered. The integration over y in (A.3) is most conveniently performed first; since ϕ_1 is an odd function of y, we have

$$\int_{-\infty}^{\infty} \omega\,dy = \chi_1(x)\cos 2\pi[\zeta_1(x)+nt] - \chi_2(x)\sin 2\pi[\zeta_1(x)+nt],$$

where $$\chi_1(x) = \int_{-5}^{5} \alpha\,dy, \qquad \chi_2(x) = \int_{-5}^{5} \beta\,dy,$$

and so

$$-\iint v\,dxdy = \left\{\int_0^\infty x\chi_1(x)\cos 2\pi\zeta_1(x)\,dx - \int_0^\infty x\chi_2(x)\sin 2\pi\zeta_1(x)\,dx\right\}\cos 2\pi nt -$$

$$-\left\{\int_0^\infty x\chi_1(x)\sin 2\pi\zeta_1(x)\,dx + \int_0^\infty x\chi_2(x)\cos 2\pi\zeta_1(x)\,dx\right\}\sin 2\pi nt.$$

An integration by parts makes these integrals more rapidly convergent;

$$-\iint v\,dxdy = \cos 2\pi nt\int_0^\infty [\theta_1(x)\sin 2\pi\zeta_1(x) + \theta_2(x)\cos 2\pi\zeta_1(x)]\,dx +$$

$$+\sin 2\pi nt\int_0^\infty [\theta_1(x)\cos 2\pi\zeta_1(x) + \theta_2(x)\sin 2\pi\zeta_1(x)]\,dx, \quad \text{(A.5)}$$

where $$\theta_1(x) = \frac{d}{dx}\left\{\frac{x\chi_1(x)}{2\pi\zeta_1'(x)}\right\},$$

with a similar expression for $\theta_2(x)$.

The expression (A.5) is that from which the fluctuations in wake momentum were calculated. Kovasznay's (1949) experimental results for $\phi_1(x,y)$ were smoothed and interpolations were made so that a map of the function was obtained for the domain $-5 < y < 5$, $0 < x < 40$. The functions $\alpha(x,y)$ and $\beta(x,y)$ were found numerically, and $\chi_1(x)$ and $\chi_2(x)$ calculated by integration. $\theta_1(x)$ and $\theta_2(x)$ were thence obtained, and the final result emerged from the integration of (A.5). It was that

$$\iint v\,dxdy = 0{\cdot}48\,Ud^2\cos 2\pi nt, \tag{A.6}$$

giving the fluctuating lift as

$$f_l = 0{\cdot}38\rho U^2 d\cos 2\pi nt, \tag{A.7}$$

since at this Reynolds number $n = 0{\cdot}13\,U/d$. These are the results quoted in (3.4) and (3.5).

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